Assignment #2

Due March 2, 2017

Christophe Audouze & Geoffrey Scott

This assignment has three problems:

1. The first problem studies how using multiple bases for the same vector space can help us understand recurrence relations that appear in biological population dynamics and recursive programming algorithms.
2. In the second problem, we use linear systems of equations to describe a configuration of masses and springs. This problem will help you gain a physical intuition for why some real-world systems have zero solutions, others have one, and others have infinitely many.
3. The third problem introduces the notion of a *Markov chain*. In it, we use concepts from linear algebra to study long-term behavior of systems appearing in economics, chemistry, and computer science.

Submission guidelines

- Submit your solutions in class, and your documented MATLAB scripts (and a file named *Readme* outlining how scripts should be run) as a **single tar or zip file** to Blackboard.

1 Recurrence Relations

Let $\mathbb{C}^\infty$ denote the vector space of all infinite sequences $(a_1, a_2, \ldots, a_n, \ldots)$ of complex numbers, with the field $\mathbb{C}$, and the usual addition and scalar multiplication of sequences. Let $Fib \subseteq \mathbb{C}^\infty$ denote the subset of all sequences that satisfy $a_{n+2} = a_n + a_{n+1}$ for all n.[1]

a. Prove that Fib is a subspace of $\mathbb{C}^\infty$.

The **Fibonacci** sequence is the element of Fib with $a_1 = a_2 = 1$. The first few terms are: $1, 1, 2, 3, 5, 8, 13, \ldots$.

b. Let

$$f_{(1,0)} = (1, 0, 1, 1, 2, 3, 5, 8, \ldots) \quad \text{and} \quad f_{(0,1)} = (0, 1, 1, 2, 3, 5, 8, 13, \ldots).$$

Prove that $\{f_{(1,0)}, f_{(0,1)}\}$ is a basis for Fib, and calculate the coordinates of the Fibonacci sequence in this basis.

c. Find two different nonzero numbers $r, s \in \mathbb{C}$ for which

$$f_{(r,r^2)} = (r, r^2, r^3, r^4, r^5, \ldots) \quad \text{and} \quad f_{(s,s^2)} = (s, s^2, s^3, s^4, s^5, \ldots)$$

are elements of Fib. Explain why $\{f_{(r,r^2)}, f_{(s,s^2)}\}$ is also a basis for Fib, and write an expression for the n^{th} term of the sequence $c_1 f_{(r,r^2)} + c_2 f_{(s,s^2)}$, where $c_1, c_2 \in \mathbb{C}$ are arbitrary constants.

Remark: We found two useful bases for the vector space Fib:

- The coordinates of an element of Fib using the basis $\{f_{(1,0)}, f_{(0,1)}\}$ describe the element's first two terms.
- The coordinates of an element of Fib using the basis $\{f_{(r,r^2)}, f_{(s,s^2)}\}$ make it easy to write an expression for the n^{th} term of the sequence.

In the next questions, we'll find the *change of basis* matrices to convert between these coordinates.

[1] The equation $a_{n+2} = a_n + a_{n+1}$ is an example of a *recurrence relation*: an equation that expresses one term in a sequence as a function of previous terms.

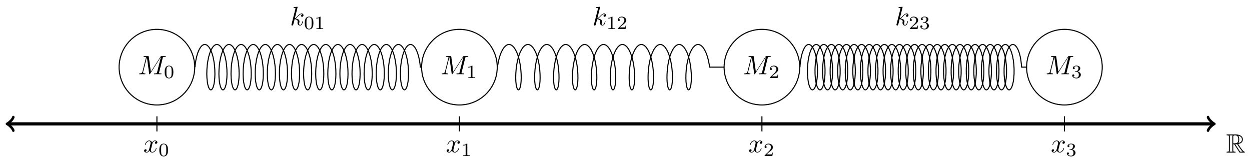
d. Calculate the change of basis matrix from the basis $\{f_{(r,r^2)}, f_{(s,s^2)}\}$ to $\{f_{(1,0)}, f_{(0,1)}\}$. Then, invert it to find the change of basis matrix from $\{f_{(1,0)}, f_{(0,1)}\}$ to $\{f_{(r,r^2)}, f_{(s,s^2)}\}$.

e. Use your answer to parts (d.) and (b.) to calculate the coordinates of the Fibonacci sequence in the basis $\{f_{(r,r^2)}, f_{(s,s^2)}\}$. Then use your answer to part (c.) to find an expression for the n^{th} term of the Fibonacci sequence.

f. For $b, c \in \mathbb{C}$, let $Fib_{(b,c)} \subseteq \mathbb{C}^\infty$ denote the set of all sequences for which $a_{n+2} = c \cdot a_n + b \cdot a_{n+1}$. Assuming that $b^2 + 4c \neq 0$, repeat the steps above to find an expression for the n^{th} term of the element in $Fib_{(b,c)}$ that begins with $(a_1, a_2, \dots) = (1, 1, \dots)$.

2 Systems of Springs

Consider the arrangement of springs and masses shown below.



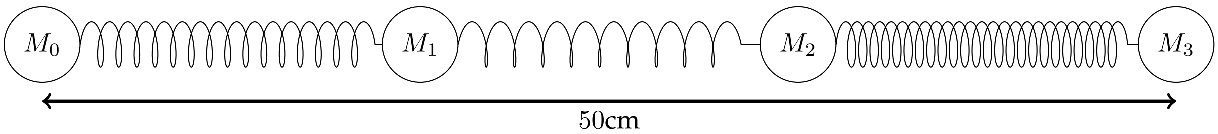
Here, the notation x_i denotes the position of the mass M_i, and k_{ij} is a constant representing the stiffness of the spring connecting M_i to M_j. For this problem, let $k_{01} = 2, k_{12} = 1$, and $k_{23} = 3$. Each spring is 10cm at rest. When a spring is extended or compressed, it exerts a force proportional to this displacement with constant of proportionality k_{ij}. For example, the spring between M_2 and M_3 exerts a force of $k_{23}(x_3 - x_2 - 10)$ on M_2 (in the rightward direction) and M_3 (in the leftward direction). Let **x** denote the column matrix $(x_0, x_1, x_2, x_3)^T$.

a. Find a matrix A and a column matrix b so that solutions of $A\mathbf{x} = b$ represent configurations for which the net force on each mass is zero. Find a basis for null(A), and also find one solution v to $A\mathbf{x} = b$ (if it exists). Remember that set of all solutions is

$$\{v + n \mid n \in \text{null}(A)\}$$

Use this fact to write a short description[2] of what the physical configurations of masses corresponding to these solutions look like.

b. Now suppose we hold mass M_0 and M_3 in our hands and pull until M_0 and M_3 are exactly 50cm apart. Find



A and b such that solutions of $A\mathbf{x} = b$ represent configurations where the net force on both M_1 and M_2 is zero, and the mass M_3 is 50cm to the right of M_0.[3]. Find a basis for null(A), find any one solution to $A\mathbf{x} = b$ (if it exists), and write a short description of what the physical configurations corresponding to solutions of $A\mathbf{x} = b$ look like.

c. Suppose you hold mass M_0 and M_3 in your hands. You hold M_0 at the point $x_0 = 0$, and you push M_3 towards it until $x_1 = 1$. Find A and b such that solutions of $A\mathbf{x} = b$ represent configurations satisfying these conditions. Find a basis for null(A), find any one solution to $A\mathbf{x} = b$ (if it exists), and write a short description of what the physical configurations corresponding to solutions of $A\mathbf{x} = b$ look like.

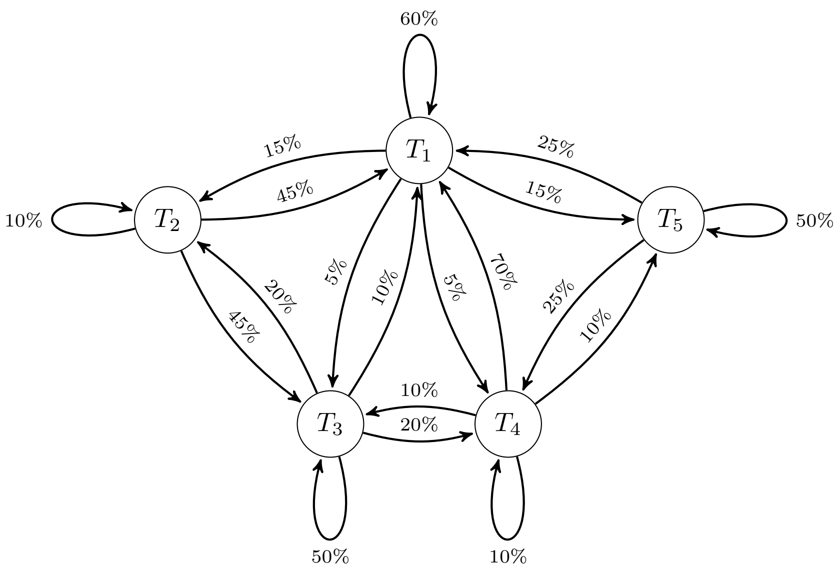
d. Suppose you hold mass M_0 and M_3 in your hands, and you position your hands so that $x_1 = 2, x_2 = 14$, and $x_3 = 25$. Find A and b such that solutions of $A\mathbf{x} = b$ represent configurations of masses satisfying these conditions. Find a basis for null(A), find any one solution to $A\mathbf{x} = b$ (if it exists), and write a short description of what the physical configurations corresponding to solutions of $A\mathbf{x} = b$ look like.

[2] A *short description* should be between one and three sentences

[3] Notice that we no longer insist that the net force due to springs on M_0 and M_3 is zero, because our hands counteract the (nonzero) force exerted by the springs on M_0 and M_3.

3 Markov Chains

Suppose you have five tanks of water $T_1, \ldots, T_5$, some of which are connected by pipes. You have a pet fish[4] which, once per minute, decides to either stay still or move to a connected tank. Her behavior is probabilistic: if she is in tank T_1, then one minute later there is a 60% chance that she will still be in tank T_1, and a $15\%, 5\%, 5\%$, and 15% chance she will be in tank T_2, T_3, T_4, or T_5, respectively. The probabilities for the other pairs of tanks is shown in the diagram below. Some tanks have no arrows between them, because these tanks are not connected by pipes.



Let P be the matrix whose (i, j) entry, p_{ij} is the probability that if your fish is in tank T_j, then it will be in tank T_i one minute later. For example, $p_{21} = 0.15$. A fact from probability theory (called the *Law of Total Probability*) implies the following.

<u>**Fact:**</u> For $t \geq 0$, the probability that your fish is found in tank T_i at time $t + 1$ is equal to

$$\sum_{j=1}^{5} \left(\begin{array}{c} \text{probability that the fish} \\ \text{is in tank } T_j \text{ at time } t \end{array} \right) \cdot p_{ij}$$

a. Use mathematical induction and the fact in the box above to prove that the (i, j) entry of P^n equals the probability that if your fish is in tank T_j at $t = 0$, then it will be in tank T_i at $t = n$.

b. Suppose your fish is in T_1 at $t = 0$. Calculate the probability that it will be in tank T_i after 2 minutes, for each T_i. Repeat the question for 50 minutes. Use MATLAB to do your calculations, but do not submit your MATLAB script for grading.

c. Calculate the matrix P^{50}. Use MATLAB, but do not submit your MATLAB script for grading. What striking fact do you observe about the columns of this matrix? Briefly interpret this observation in the context of the motion of your fish.

This model of fish movement is an example of something called a *Markov Chain*. Informally speaking, a Markov chain is a collection of *states* together with *transition probabilities* between the states. For (a non-fish) example, you could model socioeconomic mobility in society by representing socioeconomic classes as *states*, and representing the likelihood that the child of a person in socioeconomic class j ends up in socioeconomic class i by a transition probability from state j to state i. The most important theorem about the long-term behavior of such systems

[4] If you prefer physics to zoology, you can think of the tanks being filled with gas, and the fish as being a particle of gas

is the *Fundamental Theorem of Markov Chains*, which states that if P is the matrix of transition probabilities of a Markov chain, and the Markov chain satisfies some mild conditions[5] (which are satisfied by our example), then:

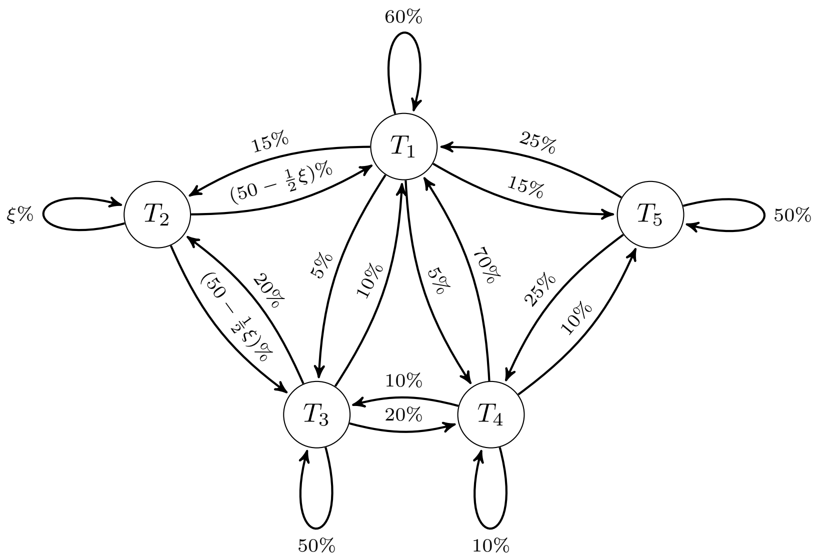
- There is exactly one vector $\mathbf{x}$ such that $P\mathbf{x} = \mathbf{x}$ whose components sum to 1. This $\mathbf{x}$ is called the *stationary distribution*.
- As $n \to \infty$, the matrix P^n approaches the matrix with the stationary distribution in each column.

Note that the matrix of transition probabilities of any Markov chain will have the property that the entries of each column sum to 1. We will prove part of the Fundamental theorem, that any such matrix has at least one solution to $P\mathbf{x} = \mathbf{x}$.

d. Let A be any $n \times n$ square matrix with the property that the elements of any column sum to 1.

- Show that null$(A^T - I)$ contains a nonzero element by finding a specific example.
- Prove that rank$(A^T - I) < n$ and rank$(A - I) < n$.
- Prove that there is a solution to the equation $A\mathbf{x} = \mathbf{x}$.

The other parts of the fundamental theorem are hard to prove. Let us accept that it is true, and return to our aquarium. Suppose you put some pleasant decorations in tank T_2. This causes your transition probabilities to change to the following, where $10 \le \xi \le 95$ is a constant that depends on how much effort you put into your decorations.[6]



e. Write a MATLAB script which, for each integer value $10 \le \xi \le 95$, calculates the second component of the stationary distribution of the Markov chain. This number represents the proportion of time that the fish will spend in tank T_2. Plot your answer as a function of ξ. Your script should *not* calculate a high power of P for each value of ξ – this would be computationally inefficient[7]! Instead it should calculate the stationary distribution for each value of ξ.

f. Use your answer to part (e) to estimate how much effort you should put into your decorations so that your fish will spend 50% of her time in tank T_2.

For any square matrix A, a nonzero solution to the equation $A\mathbf{x} = \mathbf{x}$ is called an *eigenvector with eigenvalue 1*. More generally, if λ is any scalar, a nonzero solution to $A\mathbf{x} = \lambda x$ is called an *eigenvector with eigenvalue* λ. In this problem, we saw how eigenvectors arise when studying the long-term behavior of Markov chains. In class, we'll study eigenvectors and eigenvalues in detail.

[5]The Markov chain must be "irreducible" and "aperiodic". I won't define these terms, but the specific Markov chain that we're studying in this problem satisfies these two conditions.

[6]For any $10 \le \xi \le 95$, the new Markov chain will still satisfy the conditions of the Fundamental Theorem of Markov Chains

[7]But calculating a high power of P once for some value of ξ, and comparing the second row to your answer might help check that your code works.