Algebra Homework 7

Due by the *start* of class on Mon Dec. 7

Problem 1:

1. Prove that every subgroup of a solvable group is solvable.
2. Use the fact that S_5 is not solvable to prove that S_n is not solvable for $n \geq 5$

Problem 2: Let $\alpha = \sqrt[5]{1 + \sqrt[3]{5} + \sqrt[4]{3}}$. Show that α is contained in some field of the form $\mathbb{Q}(a_1, \ldots, a_n)$, where the elements a_i have the property that there are positive integers $k_1, \ldots, k_n$ such that

$$\begin{aligned} a_1^{k_1} &\in \mathbb{Q} \\ a_2^{k_2} &\in \mathbb{Q}(a_1) \\ a_3^{k_3} &\in \mathbb{Q}(a_1, a_2) \\ &\vdots \\ a_n^{k_n} &\in \mathbb{Q}(a_1, a_2, \ldots, a_{n-1}) \end{aligned}$$

Commentary: This problem should help you reconcile the strange-looking definition of "solvable by radicals" with the intuitive concept that you had of a polynomial being "solvable by radicals" before you stepped into MAT401. After solving this problem, you should understand how any α which is some complicated nested composition of roots can be viewed as an element of a field of the kind above.

Problem 3: Suppose there is a sequence of field extensions with characteristic zero

$$F = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_n = E$$

with the property that each F_{i+1} is a splitting field over F_i, E is a splitting field over each F_i, and each $[F_{i+1} : F_i]$ is a prime number. Prove that the Galois group $\text{Gal}(E/F)$ is solvable. You may use the fact that every group whose order is a prime number is Abelian.

Problem 4: In class, we defined the *characteristic* of a field; the definition of the *characteristic* of a ring with unity is the same. That is, the *characteristic* of a ring with unity R is the smallest positive integer k such that

$$\underbrace{1 + 1 + \cdots + 1}_{k \text{ copies of } 1} = 0.$$

If no such k exists, the characteristic is zero. Let m, n be positive integers. Find the characteristic of $\mathbb{Z}_m \oplus \mathbb{Z}_n$.

Problem 5:

1. Prove that every maximal ideal in an integral domain is also a prime ideal.
2. Find an example of an ideal in an integral domain which is prime but not maximal.

Commentary: This problem and the next one are just practice for the final, to remind you of some concepts from the first parts of the course.

Problem 6: Let $F \subseteq K \subseteq E$ be fields, and $\alpha \in E$ be algebraic over F. Either prove that $[F(\alpha) : F] \geq [K(\alpha) : K]$, or find an example of F, K, E and α for which this is not true.