Algebra Homework 6

Due by the *start* of class on Wednesday Nov. 25

Problem 1:

1. Find the minimal polynomial for $\sqrt{3}+\sqrt{5}$ in $\mathbb{Q}[x]$.
2. Find the minimal polynomial for $\sqrt{3}+\sqrt{5}$ in $\mathbb{Q}(\sqrt{5})[x]$.
3. Find the minimal polynomial for $\sqrt{3}+\sqrt{5}$ in $\mathbb{Q}(\sqrt{10})[x]$.
4. Find the minimal polynomial for $\sqrt{3}+\sqrt{5}$ in $\mathbb{Q}(\sqrt{15})[x]$.

Commentary: The moral of this exercise is that whenever you have an element α of a field, the answer to the question "What is the minimal polynomial of α?" depends on exactly which polynomial ring you're considering. This should remind you of the concept of a polynomial being irreducible: a single polynomial like x^2+1 might be irreducible in some polynomial rings (e.g. $\mathbb{Q}[x]$) and reducible in others (e.g. $\mathbb{C}[x]$). So it doesn't make sense to ask "Is x^2+1 irreducible?" until you specify the polynomial ring you're considering. Likewise, it doesn't make sense to ask "What is the minimal polynomial of α?" until you specify the polynomial ring you're looking for the minimal polynomial in.

Problem 2: Prove that there are exactly two $\mathbb{R}$-automorphisms of $\mathbb{C}$: the identity map and the complex conjugation map – there are no others.

Commentary: This shows that $\text{Gal}(\mathbb{C}/\mathbb{R})$ has exactly two elements, the identity automorphism and the complex conjugation automorphism.

Problem 3:

1. Describe the Galois group of the splitting field of x^4+1 over $\mathbb{Q}$.
2. Describe the Galois group of the splitting field of x^2+9 over $\mathbb{Q}$.
3. Describe the Galois group of the splitting field of $x^2-10x+21$ over $\mathbb{Q}$.
4. Let $E=\mathbb{Q}(\sqrt{3},\sqrt{5})$. Describe the Galois group $\text{Gal}(E/\mathbb{Q})$.

Problem 4: Let $f(x)\in F[x]$ be a polynomial of degree n, and E be the splitting field for f over F.

1. Show that $[E:F]\leq n!$. (The exclamation point denotes factorial. Hint: review the proof of the construction of the splitting field)
2. Prove that $\text{Gal}(E/F)$ is isomorphic to some subgroup of the symmetric group S_n. (Hint: what is the maximum number of roots that F could have in E?)

Problem 5: Let E be the splitting field of $x^6-8\in\mathbb{Q}[x]$ over $\mathbb{Q}$. Find two roots a,b of x^6-8 such that no element of $\text{Gal}(E/\mathbb{Q})$ sends a to b. Justify your answer. (Hint: x^4+2x^2+4 divides x^6-8.)

Commentary: The reason I assigned this problem is that many students make the mistake of thinking that for a splitting field E of some polynomial over $\mathbb{Q}$, the group $\text{Gal}(E/\mathbb{Q})$ can be thought of as all possible permutations of the roots of the polynomial (in this case, roots of x^6-8). While it's good to think of elements of the Galois group in terms of how they permute the roots of the polynomial, make sure you don't think that *every* permutation of the roots is realized by some element of the Galois group.