Algebra Homework 2

Due by the *start* of class on Wednesday Sept. 30

The problems with the asterisks $*$ might need material from Wednesday's class.

Problem 1: List all of the ideals in $\mathbb{Z}_{12}$. For each ideal I that you find, what familiar ring is $\mathbb{Z}_{12}/I$ isomorphic to? You do not need to show your work.

Problem 2: Find the multiplicative inverse of $[2]x + [3]$ in $\mathbb{Z}_5[x]/\langle [1]x^2 + [1]x + [2] \rangle$. You do not need to show your work.

Problem 3*: You are a Chinese general and your army has just come back from battle. You want to count the survivors, but this would take hours: you can tell that are somewhere between 10,000 and 15,000 of them! Instead, you tell them to form groups of 15. There are three soldiers left over. Then you tell them to form groups of 16. Again, there are three soldiers left over. Finally, you tell them to form groups of 17. Again, there are three soldiers left over. Use the chinese remainder theorem to calculate the exact number of survivors. (Hint: there is one "obvious" integer k such that $[k] = [3]$ in $\mathbb{Z}_{15}, \mathbb{Z}_{16}$, and $\mathbb{Z}_{17}$, but it is not between 10,000 and 15,000).

Problem 4: Prove that the subset $\{(a, 0) \mid a \in \mathbb{Z}\}$ is a prime ideal of $\mathbb{Z} \times \mathbb{Z}$, but is not a maximal ideal.

Problem 5: Let $\mathfrak{m}_c \subseteq \mathbb{R}[x]$ be the subset of polynomials that vanish at $c \in \mathbb{R}$. Prove that $\mathfrak{m}_c$ is a maximal ideal (hint: remember what we learned about quotients of rings by maximal ideals!)

Problem 6: Let R be a commutative ring with identity. Recall that $R[x]$ is the ring of polynomials with coefficients in R. Let $f(x) \in R[x]$ be a monic polynomial of degree $d \geq 1$ (recall that a *monic* polynomial is one where the leading term has coefficient 1, and the *degree* of a polynomial is the highest power of x that appears).

1. Show that every element of $R[x]/\langle f(x) \rangle$ can be represented by a polynomial of degree $< d$.
2. Show that if two polynomials $g, h \in R[x]$ of degree $< d$ are different, then they represent different elements of $R[x]/\langle f(x) \rangle$.

Problem 7*:

1. List all elements of the ring $\mathbb{Z}_2[x]/\langle [1]x^2 + [1]x + [1] \rangle$ and write the addition and multiplication table for this ring. (hint: do problem 6 first)
2. Prove that $\langle [1]x^2 + [1]x + [1] \rangle$ is a maximal ideal in $\mathbb{Z}_2[x]$ (hint: remember what we learned about quotients of rings by maximal ideals!)