Algebra Homework 1

Due by the *start* of class on Wednesday Sept. 23

The problems with the asterisks $*$ might need material from Wednesday's class.

Problem 1: Below are two possible addition and multiplication tables for a commutative ring. In each case, determine the following:

- What is the additive identity of the ring (the term normally called "0")?
- Does the ring has a multiplicative identity (the term normally called "1")? If so, what is it?
- What are the zero divisors of the ring?
- What are the units of the ring?

You don't need to show work.

1.

+	A	B	C	D
A	A	B	C	D
B	B	A	D	C
C	C	D	A	B
D	D	C	B	A

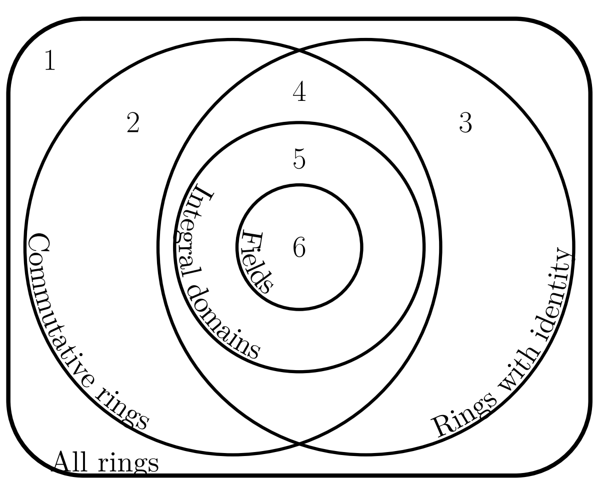
$*$	A	B	C	D
A	A	A	A	A
B	A	B	C	D
C	A	C	D	B
D	A	D	B	C

2.

+	A	B	C	D
A	A	B	C	D
B	B	A	D	C
C	C	D	A	B
D	D	C	B	A

$*$	A	B	C	D
A	A	A	A	A
B	A	B	A	B
C	A	A	C	C
D	A	B	C	D

Problem 2: Consider the following venn diagram of different kinds of rings. For each region (labeled 1-6), give an example of a ring that belongs to that region. For example, the answer to 4 should be a commutative ring with identity which is *not* an integral domain. You do not need to justify your answers or show your work.



1. (Ring that is non-commutative and has no identity)
2. (Commutative ring with no identity)
3. (Non-commutative ring with identity)
4. (Commutative ring with identity which is not an integral domain)
5. (Integral domain which is not a field)
6. (Field)

Problem 3: What are the units in the ring $\mathbb{Z}_n$, where n is any positive integer? You do not need to prove your answer.

Problem 4*: Check whether each of the following maps are homomorphisms. You do not need to show your work. In each case, if the map is a homomorphism, write down its kernel and image. Otherwise, explain (in one sentence) why the map is not a homomorphism.

1. $$\varphi : \mathbb{Z}[x] \to \mathbb{Z} \times \mathbb{Z}$$ $$a_0 + a_1 x + a_2 x^2 + \ldots \mapsto (a_0, a_1)$$

2. $$\varphi : \mathbb{Z}_2 \to \mathbb{Z}$$ $$[0] \mapsto 0$$ $$[1] \mapsto 1$$

3. $$\varphi : \mathbb{Z} \to \mathbb{Z}_6 \times \mathbb{Z}_{15}$$ $$k \mapsto ([k], [k])$$

   (the first bracket denotes the coset in $\mathbb{Z}_6$, the second denotes the coset in $\mathbb{Z}_{15}$).

Problem 5: Let a, b be elements of a ring R. Recall that the notation $-a$ means the additive inverse of a. Prove the following claims.

1. $(-a) \cdot (-b) = a \cdot b$.
2. $(-1) \cdot (a) = -a$
3. If R is a ring with identity and a is a unit, then $-a$ is also a unit.

Problem 6: An element x of a ring is called **nilpotent** if $x^n = 0$ for some n (the notation x^n is an abbreviation for x multiplied by itself n times). Let R be a ring with identity.

1. Prove that if $x \in R$ is nilpotent, then $1 + x$ is a unit. (hint: try factoring $x^n - 1$)
2. Prove that if R is commutative, then the set of nilpotent elements of R is a subring.

Problem 7: Let R be a ring. Prove that $R[x]$ is an integral domain if and only if R is an integral domain.

Problem 8*: Let R be an integral domain, and let F be its field of fractions. Prove that

$$R \to F$$
$$r \mapsto \frac{r}{1}$$

is a homomorphism, and find its kernel.