Name: ____________________________

Student Number: ____________________________

Exam 1

Math 401, Fall 2015, Geoffrey Scott

DO NOT OPEN THIS BOOKLET UNTIL YOU ARE TOLD TO DO SO

Instructions:

- Put your name and student number on this page.
- You have fifty minutes to complete the exam. I recommend planning to spend 10 minutes on each question, which leaves an extra 10 minutes for the question you get stuck on.
- You may ask Nikita questions, but he cannot answer math questions or questions like "have I shown enough work?"
- A "dictionary of ring theory" is on the next page.
- I will give partial credit for some questions, so show your work.
- You may leave early.

Grades:

Question 1: ________ (out of 8)

Question 2: ________ (out of 12)

Question 3: ________ (out of 12)

Question 4: ________ (out of 6)

Total: ________ (out of 38)

Dictionary of Ring Theory

Rings

Commutative ring: A ring R is *commutative* if $ab = ba$ for all $a, b \in R$.

Field: A commutative ring R with identity is called a *field* if every nonzero element is a unit.

Integral domain: A commutative ring R with identity is called an *integral domain* if it has no zero divisors.

Principal ideal domain: An integral domain R is a *principal ideal domain* if every ideal is principal.

Ring elements

Identity: An element a of a ring is *the identity* or *one* if $ab = ba = b$ for all $b \in R$. Note: if it exists, it is unique.

Unit: An element a of a ring with identity is a *unit* if $ab = ba = 1$ for some $b \in R$.

Zero: The additive inverse of a ring is called *zero*. Note: it is unique.

Zero divisor: An element $a \neq 0$ of a ring R is a *zero divisor* if there is $b \neq 0$ such that $ab = 0$ or $ba = 0$.

Ideals

Maximal: An ideal I of a ring R is *maximal* if there is no ideal J with $I \subsetneq J \subsetneq R$.

Prime: An ideal I is *prime* if whenever $ab \in I$, then either $a \in I$ or $b \in I$.

Principal: An ideal is *principal* if it can be generated by a single element.

Question 1: Rings

In the following two questions, a set is described with an "addition" and "multiplication" operation. In each case, state whether the given data describe a ring. If it does, describe its additive identity element and what the units are. If it does not, write (in one sentence) which ring axiom is not satisfied.

1. The set of vectors in $\mathbb{R}^3$. Addition is given by the standard formula for vector addition, and multiplication is given by the cross product of vectors. (4 points)

2. The set of all ideals in the ring $\mathbb{Z}$, with "addition" and "multiplication" operations given by the operations $I + J$ and IJ, respectively. (4 points)

Question 2: Homomorphisms

1. Let $C^\infty(\mathbb{R})$ denote the ring of functions $\mathbb{R} \to \mathbb{R}$ which are infinitely-many-times differentiable, with the standard definitions of addition and multiplication of functions (for this question, you do not need to verify that this is indeed a ring). Is the map

$$\begin{aligned} \varphi : C^\infty(\mathbb{R}) &\to \frac{\mathbb{R}[x]}{\langle x^2 \rangle} \\ f &\mapsto [f(0) + f'(0)x] \end{aligned}$$

a homomorphism, where $f'(x)$ denotes the derivative of f? Prove your answer. (6 points)

2. The map

$$\begin{aligned} \psi : \mathbb{Z}[x] &\to \mathbb{Z}_{17} \\ a_0 + a_1 x + \cdots + a_n x^n &\mapsto [a_0] \end{aligned}$$

is a homomorphism (you do not need to check this). Find a generating set for the ideal $\ker(\psi)$. You do not need to show your work. (4 points)

What familiar ring is $\mathbb{Z}[x]/\ker(\psi)$ isomorphic to? You do not need to show your work. (2 points)

Question 3: Ideals, Integral Domains, and Fields

1. Let R be an integral domain with the property that for any $x \in R$, there exists some integer $n \geq 2$ for which $x^n = x$. Prove that R is a field. Hint: $x^n = x$ can be rewritten $x^n - x = 0$. (6 points)

2. Let R be any commutative ring with unity with the property that for any $x \in R$, there exists some integer $n \geq 2$ for which $x^n = x$. Prove that every prime ideal in R is maximal. You may use the above result even if you failed to prove it. (6 points)

Question 4: Polynomials

Let $a = [1]x^4 + [2]x^2 + [4]$ and $b = [-1]x^2$ in the ring $\mathbb{Z}_2[x]/\langle [1]x^2 + [1]x + [1] \rangle$. Find a representative of the coset ab which has degree < 2. You do not need to show your work. (6 points)