UNIVERSITY OF TORONTO
Faculty of Arts and Science

DECEMBER 2015 EXAMINATIONS

MAT401H1F

Duration - 3 hours

No Aids Allowed

Name: ______________________________ Student Number: __________________

DO NOT OPEN THIS BOOKLET UNTIL YOU ARE TOLD TO DO SO

Grades:

Question 1: ________ (out of 12)

Question 2: ________ (out of 12)

Question 3: ________ (out of 15)

Question 4: ________ (out of 15)

Question 5: ________ (out of 12)

Question 6: ________ (out of 24)

Question 7: ________ (out of 12)

Question 8: ________ (out of 12)

Total: ________ (out of 114)

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Question 1

Let R be a commutative ring. For any $a \in R$, define the *annihilator* of a to be the subset

$$\mathrm{Ann}(a) = \{r \in R \mid ra = 0\}$$

1. Prove that $\mathrm{Ann}(a)$ is an ideal. (4 points)

2. Find an example of an element a of a commutative ring R for which $\{0\} \subsetneq \mathrm{Ann}(a) \subsetneq R$. (2 points)

3. Consider the homomorphism
$$\begin{aligned} \varphi : \mathbb{Z} &\to \mathbb{Z}_9 \times \mathbb{Z}_{12} \\ k &\mapsto ([k], [k]) \end{aligned}$$
(the first bracket denotes the coset in $\mathbb{Z}_9$, the second denotes the coset in $\mathbb{Z}_{12}$).

   (a) Describe the kernel of φ. (2 points)

   (b) What familiar ring is $\mathrm{im}(\varphi)$ isomorphic to? (2 points)

   (c) Find one element that is in $\mathbb{Z}_9 \times \mathbb{Z}_{12}$ but *not* in $\mathrm{im}(\varphi)$. (2 points)

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Question 2

1. Let α be a root of the irreducible polynomial $x^3 - 3x + 4 \in \mathbb{Q}[x]$. Recall from class that all elements of $F(\alpha)$ can be written in the form $a + b\alpha + c\alpha^2$, for some $a, b, c \in \mathbb{Q}$. Write α^{-1} in this form. (6 points)

2. For which values of $a \in \mathbb{Z}_5$ is $f(x) = x^2 + x + a \in \mathbb{Z}_5[x]$ irreducible in $\mathbb{Z}_5[x]$? (3 points)

3. For each of your solutions a to Part 2, how many elements does $\mathbb{Z}_5[x]/\langle x^2+x+a \rangle$ contain? (3 points)

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Question 3

Consider the polynomial ring $A = \mathbb{R}[x]$.

1. Show that the subset $S = \{f \in \mathbb{R}[x] \mid f(0) \neq 0\}$ is a multiplicatively closed subset. (4 points)

Recall that the localisation $S^{-1}A$ of A at S consists of equivalence classes of expressions of the form f/g, where $f \in \mathbb{R}[x]$ and $g \in S$. Consider the ideal $M \subseteq S^{-1}A$ given by

$$M = \left\{ \frac{f}{g} \in S^{-1}A \;\middle|\; f \in \mathbb{R}[x], g \in S, \text{ and } f(0) = 0 \right\}$$

2. Show that any element of $S^{-1}A$ which is *not* inside M is a unit. (4 points)

3. Consider the map $\phi : (S^{-1}A)/M \to \mathbb{R}$, defined by $[f/g] \mapsto f(0)/g(0)$. You can assume that ϕ is a well-defined ring homomorphism.
   (a) Show that ϕ is an isomorphism. (2 points)
   (b) Deduce that M is a maximal ideal. (2 points)
   (c) Show that M is the *unique* maximal ideal in $S^{-1}A$. (Hint: Part 2 of this problem might help). (3 points)

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Question 4

1. Define the *degree* of a field extension. Be precise, as though you were writing a textbook! (3 points)

2. Let F be a field, and suppose α, β are elements of an extension field of F. If $[F(\alpha) : F] = 12$ and $[F(\beta) : F] = 15$, what are all possible values for $[F(\alpha, \beta) : F]$? (4 points)

3. Give an example of an $\alpha \in \mathbb{C}$ such that $\mathbb{R}(\alpha)$ is transcendental, or briefly explain why such an α does not exist. (4 points)

4. Give an example of an $\alpha \in \mathbb{C}$ such that $\mathbb{Q}(\alpha)$ is transcendental, or briefly explain why such an α does not exist. (4 points)

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Question 5

1. The area of a regular pentagon of side length s is $\frac{1}{4}\sqrt{5(5+2\sqrt{5})}s^2$. If a pentagon with side length s is constructible, is it always possible to construct a square with the same area as the pentagon? Justify your answer in two or three sentences. (6 points)

2. Suppose the points $(0,0)$ and $(1,0)$ are drawn on the plane $\mathbb{R}^2$. Illustrate how to construct a point $(a,b) \in \mathbb{R}^2$ with the two properties below using a straightedge and compass, or explain why it is impossible. Your "illustration" may include several operations per picture, as long as your ideas are communicated clearly. You may use any "complicated operations" from class or homework. (6 points)

   (a) The point is on the graph of $f(x) = x^3$

   (b) The second coordinate, b, is a positive prime integer. (Remember: 1 is *not* prime)

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Question 6

Let E be the splitting field of $f(x) = x^5 - 3$ over $\mathbb{Q}$. The roots of f in $\mathbb{C}$ are $\{\sqrt[5]{3}, \sqrt[5]{3}\xi, \sqrt[5]{3}\xi^2, \sqrt[5]{3}\xi^3, \sqrt[5]{3}\xi^4\}$, where $\xi = e^{2\pi i/5}$. In this problem, you may use the fact (if you need it) that for any odd number k, the polynomial $x^{k-1} + x^{k-2} + \cdots + x + 1$ is irreducible in $\mathbb{Q}[x]$.

1. Prove that the splitting field for $x^5 - 3$ over $\mathbb{Q}$ is $\mathbb{Q}(\sqrt[5]{3}, \xi)$. (6 points)

2. Find generators of the group $\mathrm{Gal}(E/\mathbb{Q})$, and describe them in terms of how they act on the roots of f. (5 points)

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Question 6 (continued)

1. Draw a lattice diagram containing the fields $\mathbb{Q}, \mathbb{Q}(\sqrt[5]{3}), \mathbb{Q}(\xi)$, and $\mathbb{Q}(\sqrt[5]{3}, \xi)$. Label each vertical line with the degree of the corresponding field extension, and explain (in about one sentence per label) how you calculated each degree. (6 points)

2. What familiar group is $\mathrm{Gal}(K/\mathbb{Q})$ isomorphic to? Is it a subgroup of $\mathrm{Gal}(E/\mathbb{Q})$? If so, is it a *normal* subgroup? (2 points)

3. What familiar group is $\mathrm{Gal}(E/K)$ isomorphic to? Is it a subgroup of $\mathrm{Gal}(E/\mathbb{Q})$? If so, is it a *normal* subgroup? (2 points)

4. Is there an element of order 2 in $\mathrm{Gal}(E/K)$? If there is, express it in terms of the generators you found in Part 1. If there is not, briefly explain why such an element cannot exist. (3 points)

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Question 7

1. Calculate the minimal polynomial of $\alpha = \sqrt[3]{3}+(\sqrt[3]{3})^2$ over $\mathbb{Q}$. Explain why it is irreducible in one sentence. Some powers of α are written on the right. (6 points)

$$\begin{aligned}
\alpha &= 0 + 3^{1/3} + 3^{2/3} \\
\alpha^2 &= 6 + 3 \cdot 3^{1/3} + 3^{2/3} \\
\alpha^3 &= 12 + 9 \cdot 3^{1/3} + 9 \cdot 3^{2/3} \\
\alpha^4 &= 54 + 39 \cdot 3^{1/3} + 21 \cdot 3^{2/3} \\
\alpha^5 &= 180 + 117 \cdot 3^{1/3} + 93 \cdot 3^{2/3} \\
\alpha^6 &= 360 + 459 \cdot 3^{1/3} + 297 \cdot 3^{2/3} \\
&\vdots
\end{aligned}$$

2. Let E be a splitting field of some polynomial in $\mathbb{Q}[x]$. Suppose that $\mathrm{Gal}(E/\mathbb{Q})$ is isomorphic to $\mathbb{Z}_{12}$. Draw a lattice diagram of the subfields of E containing $\mathbb{Q}$, and label each vertical line with the degree of the corresponding extension. (6 points)

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Question 8

1. Let G, H be groups. Prove that the group $G \times H$ is solvable if and only if G and H are solvable groups. (6 points)

2. Recall that the splitting field K for the polynomial $3x^5 - 15x + 5$ over $\mathbb{Q}$ has a non-solvable Galois group. Let E be the splitting field of another polynomial $g \in \mathbb{Q}[x]$. Prove that if E contains K as a subfield, then g is not solvable by radicals. (6 points)

You're done with Math 401! To protect the exam, please restrict your fallen tears of joy to this box. $\longrightarrow$ $\boxed{}$