Name: ________________________

Student Number: ________________________

Exam 2

Math 401, Fall 2014, Geoffrey Scott

DO NOT OPEN THIS BOOKLET UNTIL YOU ARE TOLD TO DO SO

Instructions:

WHENEVER YOU USE THE FACT THAT A POLYNOMIAL IS IRREDUCIBLE TO ANSWER A QUESTION, EXPLAIN HOW YOU KNOW THAT THE POLYNOMIAL IS IRREDUCIBLE

- Put your name and student number on this page.
- You have fifty minutes to complete the exam. We will write the remaining time on the blackboard. I recommend planning to spend 15 minutes on each question, which leaves an extra 5 minutes for the question you get stuck on.
- You may ask Nikita or me questions, but we cannot answer math questions or questions like "have I shown enough work?"
- A "dictionary of ring theory" is not on the next page.
- I will give partial credit for some questions, so show your work.
- You may leave early.

Grades:

Question 1: ________ (out of 12)

Question 2: ________ (out of 12)

Question 3: ________ (out of 12)

Total: ________ (out of 36)

Question 1: Field Extensions: the basics

1. Let $f(x) = 2x^3 + 3x^2 + 9x + 6$, and let α be a root of $f(x)$ in some extension field of $\mathbb{Q}$. Find a basis for $\mathbb{Q}(\alpha)$ as a vector space over $\mathbb{Q}$, and calculate $[\mathbb{Q}(\alpha) : \mathbb{Q}]$. (4 points)

2. Write down the product of $(3\alpha^2)$ and $(4 + 5\alpha)$ in terms of the basis you found above. (4 points)

3. Let F be a field, and let α, β be elements of some algebraic field extension of F. Prove that if $[F(\alpha) : F]$ is coprime to $[F(\beta) : F]$, then $[F(\alpha, \beta) : F] = [F(\alpha) : F][F(\beta) : F]$. (4 points)

Question 2: Geometric Constructions

Suppose the points $(0,0)$ and $(0,1)$ are drawn on the plane $\mathbb{R}^2$. For the following three questions, either construct the objects described using any of the fundamental straightedge and compass constructions from class, or explain why it is not possible.

To show a construction, give a sequence of pictures illustrating the steps (you can do more than one step per picture, as long as the sequence is clear from the pictures). You may use the back of this page to draw constructions if you need more space.

1. The point $(\pi, 1+\pi)$ (4 points)

2. The point $(\sqrt[6]{2}, 1)$ (4 points)

3. Two points which are distance $\sqrt{3}$ apart. (4 points)

Question 3: Splitting Fields

1. Let $\xi = e^{\frac{2\pi i}{7}}$. Prove that the splitting field for $x^7 - 2$ over $\mathbb{Q}$ is $\mathbb{Q}(\sqrt[7]{2}, \xi)$. (hint: $\xi^7 = 1$). (6 points)

2. Draw a lattice containing the fields $\mathbb{Q}, \mathbb{Q}(\sqrt[7]{2}), \mathbb{Q}(\xi)$, and $\mathbb{Q}(\sqrt[7]{2}, \xi)$ with a vertical line for each field extension. Label each vertical line with the degree of the corresponding field extension, and briefly explain (in about one sentence per label) how you calculated each degree. (6 points)

   - You may assume that the result in question 1 part 3 is true, even if you personally failed to prove it.
   - You may use the fact (if you need it) that for any prime p, the polynomial $x^{p-1} + x^{p-2} + \cdots + x + 1$ is irreducible in $\mathbb{Q}[x]$