Name: ____________________________

Student Number: ____________________________

Exam 1

Math 401, Fall 2014, Geoffrey Scott

DO NOT OPEN THIS BOOKLET UNTIL YOU ARE TOLD TO DO SO

Instructions:

- Put your name and student number on this page.
- You have fifty minutes to complete the exam. We will write the remaining time on the blackboard. I recommend planning to spend 10 minutes on each question, which leaves an extra 10 minutes for the question you get stuck on.
- You may ask Nikita or me questions, but we cannot answer math questions or questions like "have I shown enough work?"
- A "dictionary of ring theory" is on the next page.
- I will give partial credit for some questions, so show your work.
- You may leave early.

Grades:

Question 1: ________ (out of 12)

Question 2: ________ (out of 12)

Question 3: ________ (out of 12)

Question 4: ________ (out of 8)

Total: ________ (out of 44)

Dictionary of Ring Theory

Rings

Commutative ring: A ring R is *commutative* if $ab = ba$ for all $a, b \in R$.

Field: A commutative ring R with identity is called a *field* if every nonzero element is a unit.

Integral domain: A commutative ring R with identity is called an *integral domain* if it has no zero divisors.

Principal ideal domain: An integral domain R is a *principal ideal domain* if every ideal is principal.

Unique factorization domain: An integral domain R is a *unique factorization domain* if every nonunit $a \in R$ can be written as a finite product of irreducible elements, and this decomposition is unique up to associates. That is, if

$$a = b_1 b_2 \dots b_j = c_1 c_2 \dots c_k$$

are two such decompositions, then $j = k$ and we can rearrange the order of the b_i's so that b_i and c_i are associates.

Ring elements

Associate: Two elements a, b of an integral domain are *associates* if $a = bc$ for some unit c.

Identity: An element a of a ring is *the identity* or *one* if $ab = ba = b$ for all $b \in R$. Note: if it exists, it is unique.

Irreducible: An element a of an integral domain is *irreducible* if whenever $a = bc$, then either b is a unit or c is a unit.

Prime: An element p of an integral domain is *prime* if it is not a unit, and whenever $p \mid ab$, then $p \mid a$ or $p \mid b$. Equivalently, p is prime iff the ideal (p) is a prime ideal.

Unit: An element a of a ring with identity is a *unit* if $ab = ba = 1$ for some $b \in R$.

Zero: The additive inverse of a ring is called *zero*. Note: it is unique.

Zero divisor: An element $a \neq 0$ of a ring R is a *zero divisor* if there is $b \neq 0$ such that $ab = 0$ or $ba = 0$.

Ideals

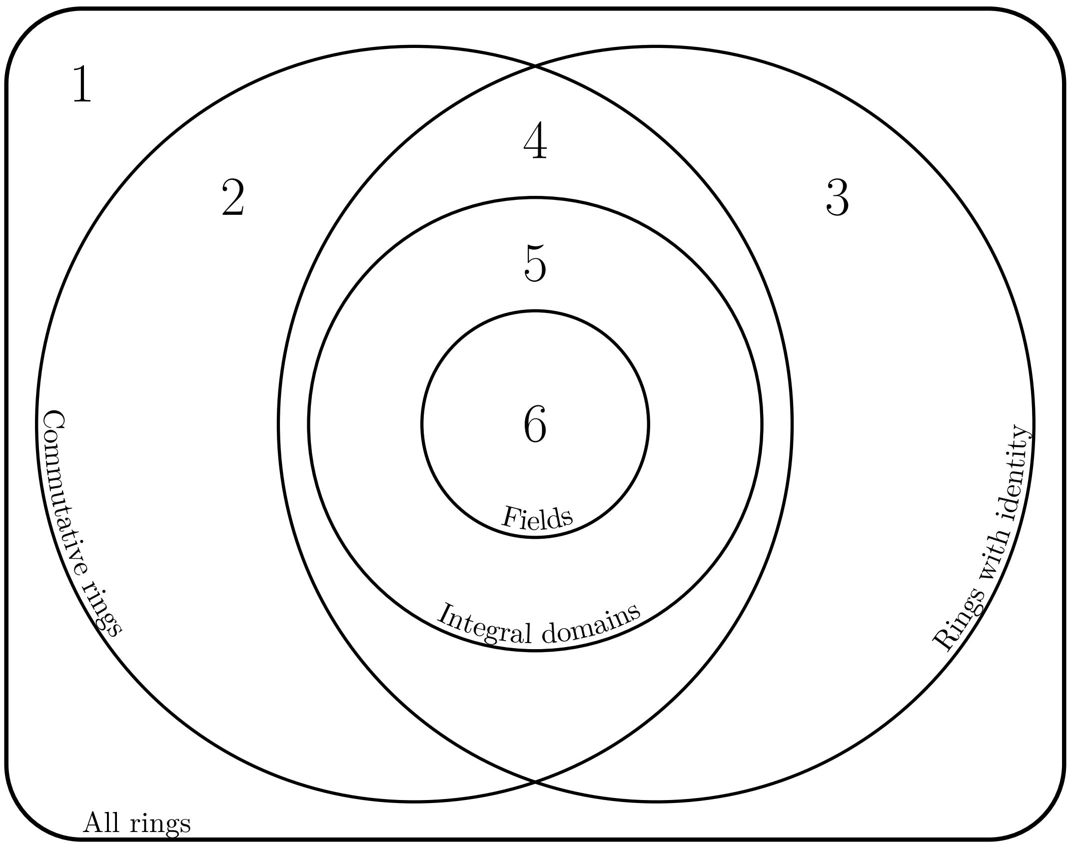
Maximal: An ideal I of a ring R is *maximal* if there is no ideal J with $I \subsetneq J \subsetneq R$.

Prime: An ideal I is *prime* if whenever $ab \in I$, then either $a \in I$ or $b \in I$.

Principal: An ideal is *principal* if it is generated by a single element.

Question 1: Rings

Consider the following venn diagram of different kinds of rings. For each region (labeled 1-6), give an example of a ring that belongs to that region. For example, the answer to 4 should be a commutative ring with identity which is *not* an integral domain. You do not need to justify your answers or show your work. (2 points each)



1. (Ring that is non-commutative and has no identity)

2. (Commutative ring with no identity)

3. (Non-commutative ring with identity)

4. (Commutative ring with identity which is not an integral domain)

5. (Integral domain which is not a field)

6. (Field)

Question 2: Homomorphisms and Ideals

1. Check whether the map

$$\varphi : \mathbb{Z}[x] \to \mathbb{Z} \times \mathbb{Z}$$
$$a_0 + a_1 x + a_2 x^2 + \ldots \mapsto (a_0, a_1)$$

   is a homomorphism. (6 points)

2. Consider the homomorphism

$$\varphi : \mathbb{Z} \to \mathbb{Z}_6 \times \mathbb{Z}_{15}$$
$$k \mapsto ([k], [k])$$

   (the first bracket denotes the coset in $\mathbb{Z}_6$, the second denotes the coset in $\mathbb{Z}_{15}$).

   (a) What familiar ring is $\mathrm{im}(\varphi)$ isomorphic to? How many elements are in $\mathrm{im}(\varphi)$? (3 points)

   (b) Find one element that is in $\mathbb{Z}_6 \times \mathbb{Z}_{15}$ but *not* in $\mathrm{im}(\varphi)$. (3 points)

Question 3: Fractions

1. Let P be a prime ideal in an integral domain R. Prove that the complement of P, $D = R \backslash P$ satisfies the following three properties: (6 points)

   (a) D is closed under multiplication

   (b) D does not contain 0

   (c) D contains no zero divisors

2. The ideal $5\mathbb{Z} \subseteq \mathbb{Z}$ is prime. Let $D = \mathbb{Z} \backslash P$, and consider the inclusions $\mathbb{Z} \subseteq D^{-1}\mathbb{Z} \subseteq \mathbb{Q}$. Give an example of an element in $\mathbb{Q}$ but not $D^{-1}\mathbb{Z}$, and an example of an element in $D^{-1}\mathbb{Z}$ but not $\mathbb{Z}$. (6 points)

Question 4: Polynomials

Let F be the field described by the following addition and multiplication tables (you verified that this was a field in homework 1)

+	0	1	a	b
0	0	1	a	b
1	1	0	b	a
a	a	b	0	1
b	b	a	1	0

*	0	1	a	b
0	0	0	0	0
1	0	1	a	b
a	0	a	b	1
b	0	b	1	a

Write $f(x) = x^4 + x^3 + ax + a \in F[x]$ as a product of irreducible polynomials. Explain (in one or two sentences) how you know that each factor in your final answer is irreducible. (8 points)