UNIVERSITY OF TORONTO
Faculty of Arts and Science

DECEMBER 2014 EXAMINATIONS

MAT401H1F

Duration - 3 hours

No Aids Allowed

Name: ______________________ Student Number: ______________

DO NOT OPEN THIS BOOKLET UNTIL YOU ARE TOLD TO DO SO

Grades:

Question 1: ________ (out of 12)

Question 2: ________ (out of 14)

Question 3: ________ (out of 16)

Question 4: ________ (out of 12)

Question 5: ________ (out of 14)

Question 6: ________ (out of 10)

Question 7: ________ (out of 16)

Question 8: ________ (out of 14)

Total: ________ (out of 108)

Name: ______________________________

Question 1

Let I be the set of polynomials in $\mathbb{Z}[x]$ that have zero constant term.

1. Verify that I is an ideal. (4 points)

2. Verify that I is a prime ideal, but not a maximal ideal. (4 points)

3. Find a homomorphism $\varphi : \mathbb{Q} \to \mathbb{Z}$ which is surjective, or prove why such a homomorphism can not exist. (Hint: What could the kernel of φ be?) (4 points)

Name: ______________________________

Question 2

1. Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$, where each $a_i \in \mathbb{Z}$ and $a_0, a_n \neq 0$. Let p/q be a root of $f(x)$, where p and q are coprime. Prove that p divides a_0, and that q divides a_n. (6 points)

2. Verify that $f(x) = x^2 + x + 2 \in \mathbb{Z}_3[x]$ is irreducible. (4 points)

3. How many elements does $\mathbb{Z}_3[x]/\langle x^2 + x + 2\rangle$ contain? (4 points)

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Question 3

1. State Eisenstein's criterion. Be precise, as though you were writing a textbook! (6 points)

2. Define the *degree* of a field extension. (4 points)

3. Let E be a field extension of F, and let $f(x)$ be an irreducible polynomial in $F[x]$ with $\deg(f(x)) \geq 2$. Prove that if $\deg(f(x))$ is coprime to $[E : F]$, then $f(x)$ has no roots in E. (Hint: If $\alpha \in E$ were a root, what could $[F(\alpha) : F]$ be?) (6 points).

Name: ______________________________

Question 4

Suppose the points $(0, 0)$ and $(1, 0)$ are drawn on the plane $\mathbb{R}^2$.

1. Construct the angle $\pi/3$ using only the three fundamental straightedge and compass operations from class. Your answer should consist of a sequence of drawings that illustrates the construction. (6 points)

2. Think of a point on $\mathbb{R}^2$ which cannot be constructed, and has coordinates that are algebraic over $\mathbb{Q}$. Give the coordinates of the point, and explain why it cannot be constructed. (6 points)

Name: ______________________________

Question 5

Let $\alpha = e^{2\pi i/n}$, where n is an integer bigger than 3.

1. Verify that $\mathbb{Q}(\alpha)$ is a splitting field over $\mathbb{Q}$. (6 points)

2. Let $n = 6$. Find two roots a, b of $x^6 - 1$ such that no element of $\mathrm{Gal}(\mathbb{Q}(\alpha)/\mathbb{Q})$ sends a to b. Justify your answer. (Hint: Some roots of $x^6 - 1$ are also roots of $x^3 - 1$, but other roots of $x^6 - 1$ are not.) (8 points)

Name: ______________________________

Question 6

Let F be a field of characteristic zero, and E be the splitting field of some polynomial in $F[x]$.

1. Prove that there are finitely many subfields of E that contain F. (4 points)

2. Suppose that $\mathrm{Gal}(E/F) \cong S_4$, and remember that S_4 has 24 elements. Determine how many subfields K of E containing F have the property that $[K : F] = 12$. (Hint: It is easier to count subgroups than subfields!) (6 points)

Name: ______________________________

Question 7

Let $E = \mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5})$.

1. Find generators of the group $\mathrm{Gal}(E/\mathbb{Q})$. (6 points)

2. What familiar group is $\mathrm{Gal}(E/\mathbb{Q})$ isomorphic to? (4 points)

3. Describe a subgroup of order four of $\mathrm{Gal}(E/\mathbb{Q})$ in terms of the generators you gave in part (1), and describe its fixed field in E. (6 points)

Name: ________________________________

Question 8

1. Recall that for $n \geq 3$, the dihedral group D_n is the group of symmetries of a regular n-gon. Prove that D_n is a solvable group. (6 points)

2. Define what it means for a polynomial $f(x)$ to be *solvable by radicals.* (4 points)

3. Let F be a field of characteristic zero, and let $f(x) \in F[x]$ be a polynomial which is solvable by radicals, and E be the splitting field for $f(x)$ over F. Give an example of a group which could *not possibly* be isomorphic to $\mathrm{Gal}(E/F)$. Briefly explain your answer in one or two sentences. (4 points)

You're done with Math 401! To protect the exam, please restrict your fallen tears of joy to this box. $\longrightarrow$ $\boxed{}$