Oct 6: Latin Rectangles

MAT332 – Geoffrey Scott

Definition: An $r \times n$ matrix consisting of the numbers $1, \ldots, n$ is called a $r \times n$ **latin rectangle** if the same number never appears twice in any row or any column. If $r = n$, it is a **latin square**.

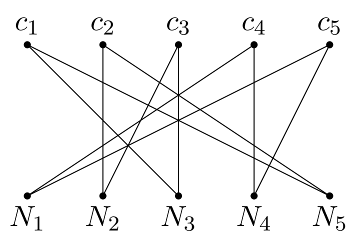
For example,

$$\begin{pmatrix} 1 & 3 & 4 & 2 & 5 \\ 4 & 1 & 5 & 3 & 2 \\ 2 & 4 & 1 & 5 & 3 \end{pmatrix}$$

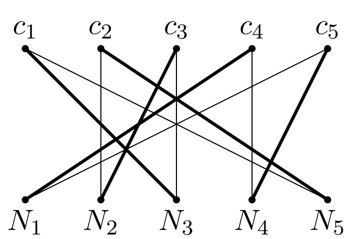
is a 3×5 latin rectangle.

Proposition 0.1. *Let $r < n$. Any $r \times n$ latin rectangle can be extended to a $(r+1) \times n$ latin square.*

Proof. Consider the bipartite graph where one of the partite sets of vertices corresponds to columns $c_1, \ldots, c_n$ of the Latin rectangle, and the other partite set of vertices $N_1, \ldots, N_n$ corresponds to the numbers $1, \ldots, n$. Put an edge between vertex c_i and vertex N_j if the column corresponding to c_i does *not* contain the number corresponding to the vertex N_j. For the example above, we have



A perfect matching of this graph corresponds to a way of assigning to each column a number which is not already present in that column. This gives a recipe for extending the latin square by one row.



$$\longrightarrow \begin{pmatrix} 1 & 3 & 4 & 2 & 5 \\ 4 & 1 & 5 & 3 & 2 \\ 2 & 4 & 1 & 5 & 3 \\ 3 & 5 & 2 & 1 & 4 \end{pmatrix}$$

To complete the proof, it suffices to verify that this graph is $(n-r)$ regular. The edges incident to the c_i vertices correpsond to numbers that *don't* appear in column c_i. There are $(n-r)$ such numbers, so $d(c_i) = n - r$. Next, notice that each number appears r times in the $r \times n$ latin rectangle because it appears once in each row. Therefore, for each number, there are exactly $n - r$ columns which *don't* have that number, so $d(N_i) = n - r$. □

By $(n-r)$ repeated applications of this proposition to an $r \times n$ latin rectangle, we arrive at a latin square, proving the following corollary.

Corollary 0.2. *Every Latin rectangle can be extended to a Latin square*