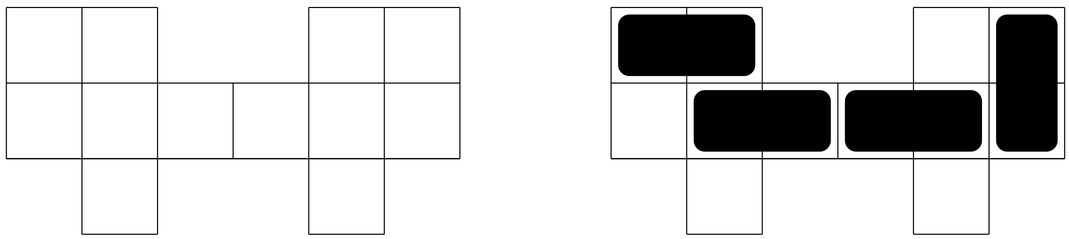
MAT332 - Fall 2016 - Homework 5

1. Consider the following board of square tiles. You want to place as many non-overlapping 2×1 tiles on the board as possible. The attempt below shows that a configuration with 4 tiles exists.



a. Draw a network G with the property that the maximal feasible flows of the network correspond to maximal ways of placing tiles on the board. (Hint: it might help to think of the squares on the board as being painted black and white in a checkerboard pattern)

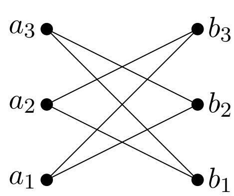
b. Give an example of a maximal flow in G, and draw the configuration of tiles that it represents.

2. A simple graph G is called k**-critical** if $\chi(G) = k$ and $\chi(H) < k$ for all proper subgraphs H of G (a subgraph is called *proper* if it is not the entire graph).

a. Let G be a k-critical graph for some $k \geq 2$. Prove that G has no vertices of degree $< k - 1$.

b. Let G be *any* simple graph, and let $k = \chi(G)$. Prove that G has at least k vertices of degree $\geq k - 1$.

3. Let G_n be the graph with vertex set $\{a_1, \ldots, a_n\}$ and $\{b_1, \ldots, b_n\}$, and an edge between a_i and b_j if and only if $i \neq j$. For example, the G_3 is shown below.



a. Describe a vertex ordering of G_n so that the greedy colouring produces a coloring of size 2.

b. Describe a vertex ordering of G_n so that the greedy coloring produces a colouring of size n.

4. Let G be a simple graph. Prove that there is some ordering of the vertices of G so that the greedy algorithm produces a vertex colouring of size $\chi(G)$.

5. You manage the office building for an energy plant that stays open at all hours of the day. You are handed a list of events that require a room. Each of these events happens *at the same time every single day*. For example:

11:30am-2:00pm A room is required for employees to eat lunch every day during this time.

4:30pm-5:00pm A room is required for a administrative meeting every day during this time.

11:00pm-1:00am A room is required for the night-shift employees to take a break every day during this time.

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You draw a graph G where each event is represented by a vertex, and two events are joined by an edge if their times overlap. You know that the number of rooms necessary to accommodate all of the events equals the chromatic number of G. Prove that $\chi(G) = \omega(G)$, or give a counterexample.

6. Let G be a simple graph with chromatic polynomial $a_0 + a_1k + a_2k^2 + \dots$. Prove that

$$\sum_{i=0}^{\infty} a_i = \begin{cases} 0 & \text{if } G \text{ has an edge} \\ 1 & \text{if } G \text{ has no edge} \end{cases}$$

7. You are the operations manager of a chemical storage facility. You are receiving a shipment of seven chemicals. Below is a list of the ingredients of each chemical.

Chemical 1: $\{A^+, B^+\}$

Chemical 2: $\{A^-, C^+\}$

Chemical 3: $\{A^-, C^-\}$

Chemical 4: $\{A^+, E^+\}$

Chemical 5: $\{B^-, D^+\}$

Chemical 6: $\{B^-, D^-\}$

Chemical 7: $\{B^+, E^-\}$

If a chemical containing the ingredient A^+ comes into contact with a chemical containing A^-, it will cause an explosion. Similarly for B^+ and B^-, C^+ and C^-, etc. For safety reasons, if two chemicals cause an explosion you must store them in separate buildings.

a. Find a graph G for which colorings of G correspond to safe ways to store the chemicals in different buildings.

b. Determine the smallest number of buildings needed to store the chemicals. Justify your answer (i.e. if you say that $\chi(G)$ is some certain number, you must explain how you know that it's that number).