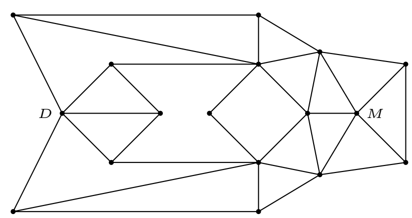
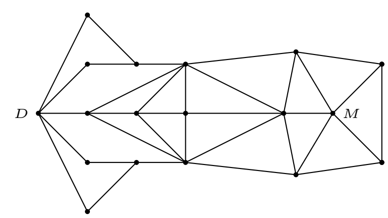
MAT332 - Fall 2016 - Homework 4

1. When the Ottoman empire built the Hejaz railway through the Arabian desert in 1908, they constantly struggled with desert tribes stealing the wooden railway ties for firewood. In addition, the tracks were bombed 30 separate times over the course of two years by Arab troops led by Lawrence of Arabia during World War I. Together, these effects made the Hejaz railway nearly as unreliable as today's TTC. You are redesigning the Hejaz. Given its history, you want to make sure that Damascus and Medina (labeled D and M, respectively) stay connected even if several tracks are out of service. Compare the two plans below.





a For each proposed railway plan, find a minimal-size set of edges that disconnects Damascus from Medina. Prove your answers are correct by finding a equal-sized set of pairwise edge-disjoint paths from Damascus to Medina. (then, the "edge" version of Menger's theorem guarantees that your set of edges really is minimal).

b Suppose you are also concerned about the security of the stations (instead of the tracks). For each proposed railway plan, find a minimal-size set of vertices that disconnects Damascus from Medina. Prove that your answers are connect by finding an equal-sized set of internally disjoint paths from Damascus to Medina.

2. Let G be a simple connected graph.

a Prove that a vertex v is contained in ≥ 2 blocks of G if and only if it is a cut vertex.

b Let $B_1, \ldots, B_k$ be the blocks of G. Prove that $|V(G)| = |V(B_1)| + \cdots + |V(B_k)| - k + 1$.

c Prove that two distinct edges belong to the same block of G if and only if there is a cycle that contains both of them.

3. The *Fifteen puzzle* is a toy consisting of fifteen square tiles numbered 1 to 15, randomly[1] arranged in a 4×4 game board, leaving the last square empty. A "legal move" consists of sliding a tile which is adjacent to the empty square into the empty square. The goal is to perform a sequence of legal moves so that the tiles are sorted.

15	6	5	12
14	7	1	9
11	2	3	10
8	13	4	

Start

15	6	5	12
14	7	1	9
11	2	3	
8	13	4	10

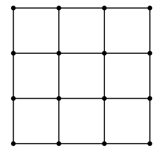
After one move

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	

Goal

If you are confused, do a google search for "fifteen puzzle" and play with an online implementation for five minutes. We can represent a configuration of a puzzle by a labeling the vertices of the graph G shown below with $\{1, 2, \ldots, 14, 15, e\}$, where "$e$" stands for the empty square, and no two vertices have the same label.

[1] It's not truly random if you want to make sure that your puzzle can be solved. But this technicality is irrelevant to us right now.

$G =$ 

Each vertex of G represents a square of the original grid, with an edge between two vertices if the squares are adjacent. A *legal move* consists of swapping the label of the "e" vertex with the label of any adjacent vertex.

We can generalize the Fifteen Puzzle to other graphs. Let G be any simple graph whose vertices are labeled $\{1, 2, \ldots, |V(G)| - 1, e\}$ such that no two vertices have the same label. A *legal move* consists of swapping the label e with the label of one of its adjacent vertices.

a Let v be a vertex in a simple 2-connected graph G, labeled as described above. Prove that it is possible to perform a sequence of legal moves so that v has the label "1".

b Let G be a simple connected graph as labeled as described above. Let v be the vertex labeled e and let w be the vertex labeled "1". Prove that it is possible to perform a sequence of legal moves so that v has label "1" if and only if there is a block of G that contains both v and w. You may use the result of problem 2 on this homework assignment if you find it helpful.

c The following is a false proof that the classic Fifteen puzzle (using the original "grid" graph) can be solved for any initial configuration. Write a few sentences (two should suffice) explaining the error in the proof. You do NOT need to prove that not every initial setup can be solved. Just find the mistake in the proof below.

The graph G corresponding to the classic Fifteen puzzle is simple and 2-connected. By part (a), it is possible to move the tile "1" to the top left square. Of course, nothing is special about the label "1" in the proof of part (a): the same result holds for any choice of label. So we can use part (a) again – it is possible to move the tile "2" to the top row, second column. Keep repeating. For each number from 1 to 15, we can move the tile to its position in the "goal" configuration. This solves the puzzle.

4. Let G be a k-connected graph, and let S, T be disjoint subsets of the vertices of G, with $|S| \geq k$ and $|T| \geq k$. Prove that there exist k pairwise disjoint paths starting from a vertex in S and ending at a vertex in T (we say that paths are *pairwise disjoint* if no vertex of G is part of more than one path).

5. Let f be a feasible flow in a network. Recall that the *value* of the flow is defined as $\text{val}(f) = f^-(t) - f^+(t)$. Prove that $\text{val}(f) = f^+(s) - f^-(s)$.

6. Find a maximum flow from s to t. Prove that your answer is optimal using the min cut max flow theorem.