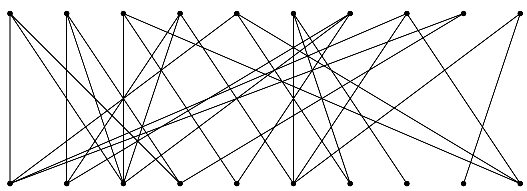
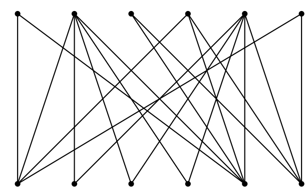
MAT332 - Fall 2016 - Homework 3

1. For the following bipartite graphs, give an example of a matching of maximum possible size, and an example of a vertex cover of minimum possible size. Explain how you can be sure that your matching really does have maximum possible size, and your vertex cover really does have minimum possible size (using a result from class, this explanation should take exactly one sentence).





2. Let G be a bipartite graph with bipartition $V(G) = A \sqcup B$ such that every vertex in A has degree n and every vertex in B has degree m, where n, m are both ≥ 1. Give an example of such a G for which $m \neq n$, or prove that no such graph exists.

3. Consider the preference list on $K_{6,6}$ below.

$a_1 : b_1 > b_2 > b_4 > b_3 > b_6 > b_5$	$b_1 : a_6 > a_4 > a_5 > a_1 > a_2 > a_3$
$a_2 : b_1 > b_2 > b_3 > b_6 > b_5 > b_4$	$b_2 : a_5 > a_6 > a_3 > a_4 > a_2 > a_1$
$a_3 : b_3 > b_2 > b_4 > b_1 > b_6 > b_5$	$b_3 : a_2 > a_4 > a_3 > a_5 > a_1 > a_6$
$a_4 : b_3 > b_1 > b_4 > b_2 > b_5 > b_6$	$b_4 : a_3 > a_5 > a_1 > a_4 > a_6 > a_2$
$a_5 : b_3 > b_4 > b_1 > b_2 > b_6 > b_5$	$b_5 : a_1 > a_2 > a_4 > a_3 > a_5 > a_6$
$a_6 : b_4 > b_5 > b_6 > b_3 > b_2 > b_1$	$b_6 : a_1 > a_3 > a_4 > a_2 > a_6 > a_5$

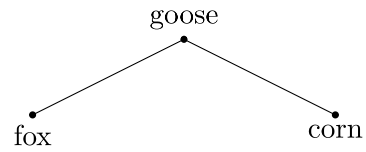
a. Determine the stable matching produced by the Gale-Shapley algorithm if the a vertices do the proposing.

b. Determine the stable matching produced by the Gale-Shapley algorithm if the b vertices do the proposing.

4. Consider a preference list on the complete bipartite graph $K_{n,n}$. The vertices in one partite set correspond to n men, and the vertices in the other partite set correspond to n women. For some $k < n$, exactly k men have green hair, and exactly k women have green hair. Suppose all people think that green hair is remarkably attractive, so on every person's preference list, every green-haired person occurs before every non-green-haired person. Prove that in every stable matching, every green-hair person is matched to another green-haired person.

5. Consider the problem of transporting n items across a river using a boat that can carry k items (besides yourself). Some pairs of items cannot be left together unsupervised. For example, perhaps one item is a goose and another is a bag of corn, which will be eaten by the goose unless you're there to prevent it. Let G be the simple graph whose vertices correspond to items, and whose edges join pairs of items that require supervision.

We wish to study the question of determining the smallest-sized boat for which transportation is possible. For example, it is possible to transport the items in the graph below using a boat that can hold $k = 1$ items by crossing with the goose, returning alone,



crossing with the fox, returning with the goose, crossing with the corn, returning alone, then crossing with the goose.

However, if you added an edge between the fox and the corn, then you would need a size $k = 2$ boat.

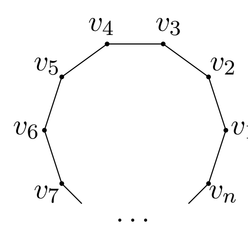
a. Prove that the minimum size of a boat k for which a solution exists is either $vc(G)$ or $vc(G) + 1$, where $vc(G)$ denotes the minimum size of a vertex cover of G.

b. Give an example of a connected graph G with 5 vertices for which a boat of size $vc(G)$ suffices to transport the items.

c. Give an example of a connected graph G with 5 vertices for which a boat of size $vc(G) + 1$ is required to transport the items.

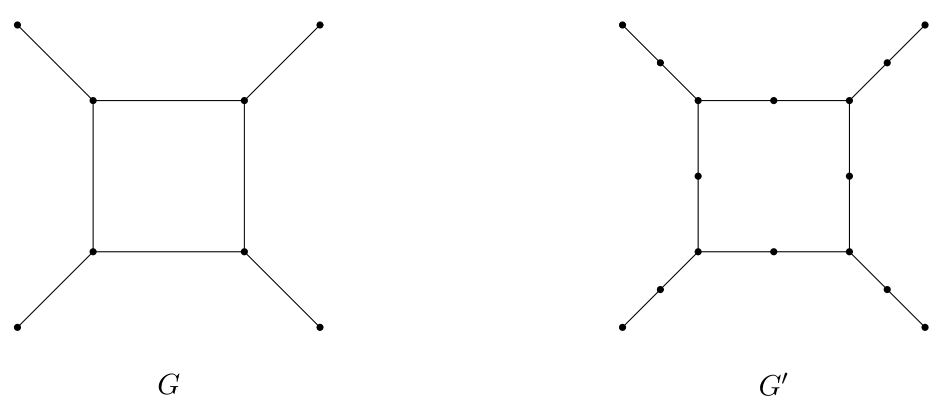
6. Let G be a connected simple graph with the property that $|E(G)| = |V(G)|$.

a. Prove that if G has no leaf, then G is isomorphic to C_n for some n, where C_n is the graph below.



b. Prove that every subgraph H of G has the property that $|E(H)| \leq |V(H)|$.

c. Let G' be the graph obtained by bisecting each edge. That is, for each edge e with endpoints v, w, introduce a new vertex v_e, and replace the edge e with two edges: one joining joining v to v_e and one joining v_e to w. The figure below shows one possible pair G, G'.



For any connected simple G with $|V(G)| = |E(G)|$, use Hall's theorem to prove that G' has a perfect matching.