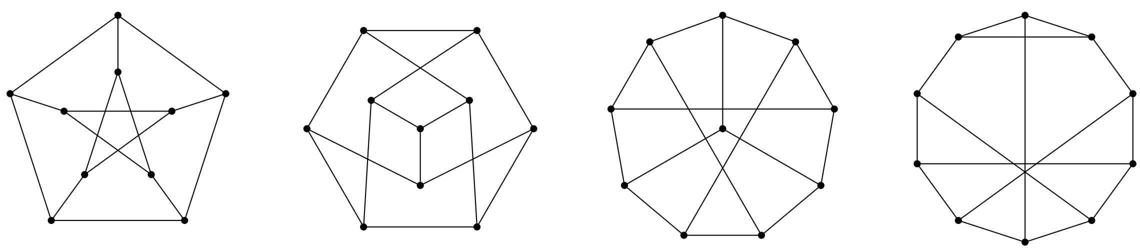
MAT332 - Fall 2016 - Homework 2

Instructor Geoffrey Scott

1. Consider the following four graphs

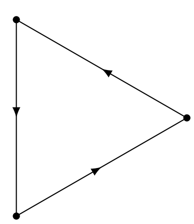


Graph 1 Graph 2 Graph 3 Graph 4

a. Prove that Graphs 1, 2, and 3 are all isomorphic to one another.

b. Prove that Graph 4 is *not* isomorphic to any of Graphs 1, 2, or 3.

2. Prove that a tournament T has more than one Hamiltonian path if and only if it has a subgraph isomorphic to the tournament RPS shown below.



The RPS Tournament

3. Let $n \geq 2$, and let $(d_1, \ldots, d_n)$ be a sequence of non-increasing integers. Prove that there is a tree with degree sequence $(d_1, \ldots, d_n)$ if and only if each $d_i \geq 1$ and $\sum_{i=1}^{n} d_i = 2(n-1)$. (Hint: induction)

4. For a graph G, an **independent set** of vertices is a subset $A \subseteq V(G)$ for which no edge of G has both of its endpoints in A. A **partition** of a set S is a collection of disjoint subsets of S whose union is all of S.

a. Prove that every tree is bipartite.

b. For a forest F with n vertices and k connected components, how many ways are there to partition $V(F)$ into two independent sets? You do not need to justify your answer.

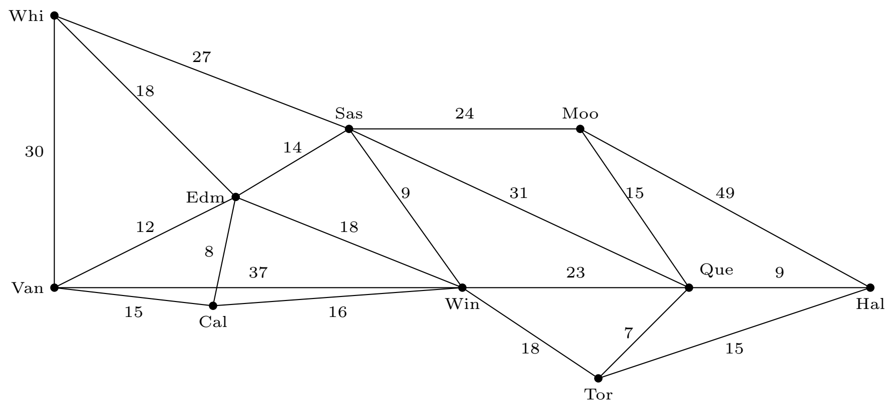
5. Let T be a tree on n vertices with degree sequence $(d_1, d_2, \ldots, d_n)$. Prove that G has at least d_1 leaves. (Remember: according to our convention, the numbers in a degree sequence are always written so that $d_1 \geq d_2 \geq d_3 \geq \ldots \geq d_n$).

6. Draw the tree with Prüfer code $(2, 1, 5, 5, 7, 7)$. Remember that the Prüfer code encodes trees whose vertices have been labeled by certain integers. Make sure to include these labels in your drawing.

7. Consider the weighted graph of Canadian cities shown below.

a. If you were to apply Kruskal's algorithm to find the minimal spanning tree of the graph, write down a list of the edges that you would include in the order that they are added in the algorithm, and draw the corresponding spanning tree.

b. Apply Dijkstra's algorithm to find the distance from the vertex Van to all other vertices. To show your answer, draw the spanning tree that results from Dijkstra's algorithm, and label each vertex with its distance from Van.



8. A popular children's puzzle asks you to imagine four people arriving at a bridge in the middle of the night. They have only one flashlight, and the bridge can hold at most two people at a time, one of whom must be carrying the flashlight to cross. The four people walk at different speeds, and can cross the bridge in 1 minute, 2 minutes, 5 minutes, and 8 minutes, respectively. When two people cross the bridge together, they walk at the speed of the slowest person. The puzzle asks you to find the minimal time required for all four people to cross. The answer to the puzzle is fifteen minutes. Take some time to personally convince yourself that this answer is correct. Now, describe how you could use Dijkstra's algorithm to find this answer. Do *not* draw the relevant graph or actually do the computation – just write a paragraph explaining how to express the puzzle as a minimal path problem that could be solved using Dijkstra's algorithm.

9. If G, H are graphs, the **union** of G and H, written $G \cup H$, is the graph whose vertex set is $V(G) \cup V(H)$ and whose edge set is $E(G) \cup E(H)$, and the endpoints of an edge in $G \cup H$ are the same vertices as in its original graph.

Given a cycle in a weighted graph, the **weight** of the cycle is the sum of the weights of its edges.

Consider the complete graph K_n for $n \geq 3$ whose edges have integer weights, and let H denote the spanning subgraph of K_n whose edge set consists of the edges of K_n with even weight. Prove that every cycle in this K_n has even weight if and only if H is either the entire graph K_n, or H is isomorphic to $K_\ell \cup K_m$, where ℓ, m are positive integers with $\ell + m = n$