UNIVERSITY OF TORONTO
Faculty of Arts and Science

DECEMBER 2016 EXAMINATIONS

MAT332H1F

Duration - 3 hours

No Aids Allowed

Name: ______________________ Student Number: ______________

DO NOT OPEN THIS BOOKLET UNTIL YOU ARE TOLD TO DO SO

Instructions:

- Put your name and student number on this page.
- You may ask me questions, but I cannot answer math questions or questions like "have I shown enough work?"
- I will give partial credit for some questions, so show your work.
- You may use the back of the pages as scratch paper. If you need to use the back of the pages to write your answer, make sure you write "*SEE BACK OF PAGE*" so that the grader knows where to look.
- You may leave early. Put your completed exam on the front table and enjoy your Saturday.

Grades:

Question 1: __________ (out of 120,000)

Question 2: __________ (out of 140,000)

Question 3: __________ (out of 150,000)

Question 4: __________ (out of 210,000)

Question 5: __________ (out of 130,000)

Question 6: __________ (out of 100,000)

Question 7: __________ (out of 150,000)

Total: __________ (out of 1,000,000)

Confusing Notation

Walk: A **walk** in a graph is a sequence of the form $v_0e_1v_1e_2v_2 \ldots e_nv_n$, where the v_i are vertices and the e_i are edges, such that the endpoints of e_i are v_{i-1} and v_i. A walk is **closed** if $v_0 = v_n$.

Trail: A **trail** is a walk in which no edge occurs more than once.

Path: A **path** is a trail in which no vertex occurs more than once, except that v_0 may equal v_n.

Circuit: A **circuit** is a closed trail.

Cycle: A **cycle** is a closed path.

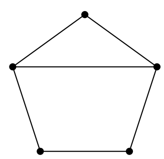
Connectivity: The **connectivity** of a graph G is the minimum size of a vertex set S such that $G - S$ is either disconnected, or has just one vertex. A graph is k**-connected** if its connectivity is $\geq k$.

Edge Connectivity: The **edge-connectivity** of a graph G is the minimum size of a set F of edges such that $G - F$ is disconnected. A graph is k**-edge-connected** if its edge-connectivity is $\geq k$.

Name: ______________________________

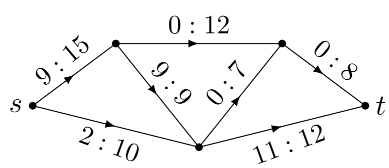
1. Algorithms

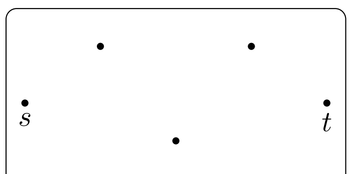
a. Calculate the chromatic polynomial of the graph below. You do not need to simplify your answer by factoring or expanding – any correct expression will receive full marks. (50,000 points)



ANSWER

b. Consider the flow f on network below. Each edge is labeled $a : b$, where a is the flow and b is the capacity. Find all f-augmenting paths from s to t. Draw them in the small boxes provided, and write the tolerance. You might not need all seven boxes. Then, pick the f-augmenting path with maximum tolerance. Draw the corresponding augmented flow f' in the large box below by labeling the edges appropriately, and write its value. (70,000 points)





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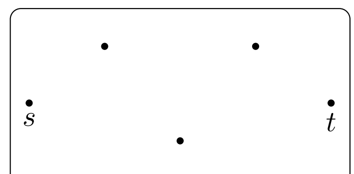

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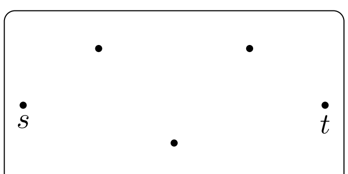

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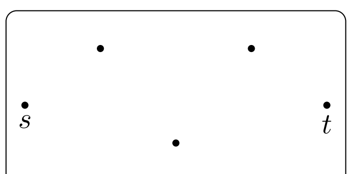

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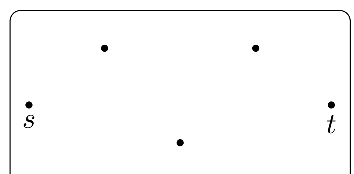

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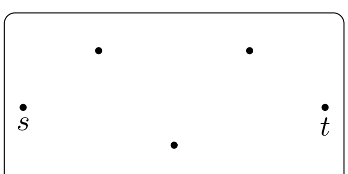

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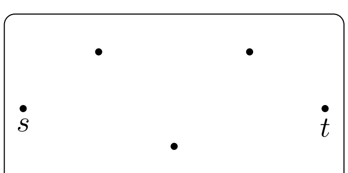

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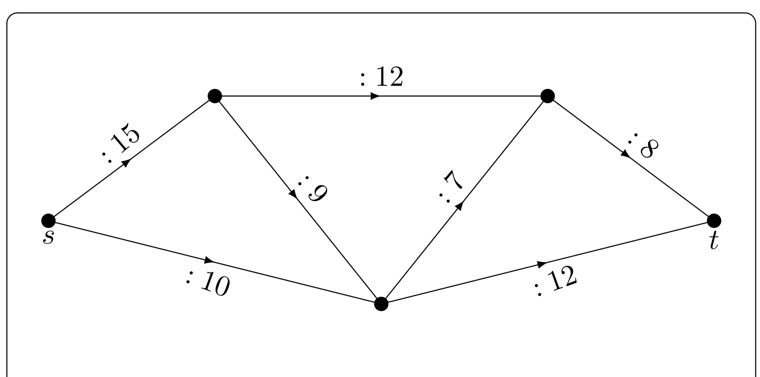


VALUE OF f' =

Name: ______________________________

2. Proofs from Class

a. Let G be an interval graph. Prove that the chromatic number of G equals the size of the largest clique in G. (60,000 points)

b. State, but do not prove, Tutte's theorem. Be precise, as though you were writing a textbook! (40,000 points)

c. State, but do not prove, Menger's theorm. Be precise, as though you were writing a textbook! (40,000 points)

Name: ______________________________

3. Short Answer

a. Let G be a forest. Give an expression in terms of $|V(G)|$ and $|E(G)|$ for the smallest number of edges you must add to G so that the result is a tree. (50,000 points)

ANSWER

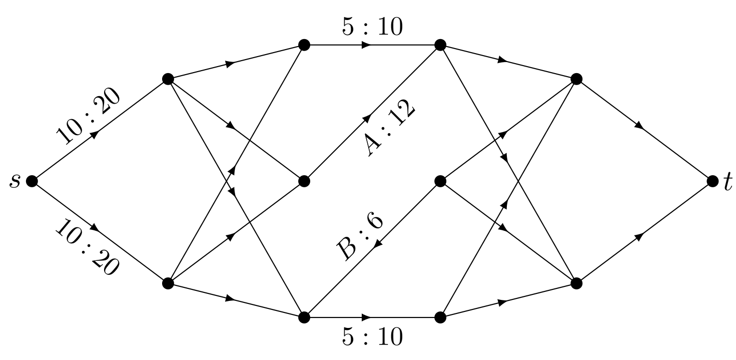
b. Suppose G is a connected planar graph with degree sequence

$$(\underbrace{3,3,3,3,3,3,3,3,3,3,3,3}_{\text{12 copies of 3}},\underbrace{2,2,2,2,2,2,2,2,2,2,2,2}_{\text{12 copies of 2}}).$$

How many faces does G have? List all possible values. (50,000 points)

ANSWER

c. Let f be a feasible flow on the network below. You know some of the values of f on the edges and some of the capacities of the edges, but not all of them. Two unknown values of f are labeled A and B. Write the range of possible values for A and B in the ANSWER boxes. For example, if you can determine that A is between 2 and 4, you should write $[2,4]$ or $2 \le A \le 4$ in the corresponding box. (50,000 points)



VALUES FOR A

VALUES FOR B

Name: ______________________________

4. Applications of graph theory

Each of the next three problems (problems **b.** and **c.** are on the next pages) can be solved by using one of the following five techniques from graph theory. Some techniques will not be used.

1. Find a minimal weight spanning tree of a weighted graph G.
2. Find a maximum size matching of a graph G.
3. Find a minimum size edge coloring of a graph G.
4. Find a maximum value flow of a network G.
5. Find a stable matching of a bipartite graph G with preference list.

For each problem, do three things. First, in the box labeled TECHNIQUE write the number of the technique above that you can use to solve the problem. Second, draw the corresponding graph G. Third, explain (in ≤ 2 sentences) how a solution to the graph theory problem corresponds to a solution to the original problem. **You do not need to actually solve the problem!** For example, if you write "2" in the TECHNIQUE box, you should draw a graph G and explain how a maximal matching in G corresponds to a solution to the problem **but you do not need to find the maximal matching**. (70,000 points each)

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a. Because of a forest fire, 20 alligators, 30 birds, 15 cats, and 20 dogs need medicine. There are three animal hospitals nearby, each with a limited amount of medicine for each type of animal. Also, each hospital has a maximum capacity of 30 animals total.

Hospital X: Can help 0 alligators, 27 birds, 12 cats, and 0 dogs.

Hospital Y: Can help 15 alligators, 17 birds, 0 cats, and 16 dogs.

Hospital Z: Can help 7 alligators, 0 birds, 12 cats, and 13 dogs.

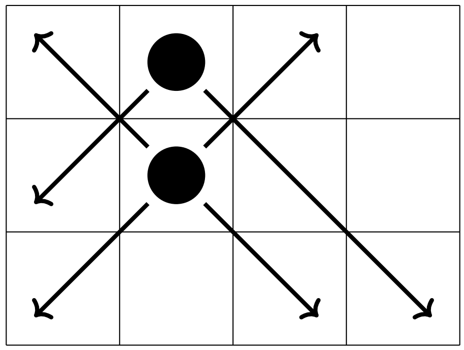
Problem: Determine how many animals of each type to send to each hospital. Your goal is to maximize the number animals helped.

TECHNIQUE

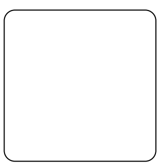
Name: ______________________________

4. Application of Graph Theory (continued)

b. Consider a 3×4 grid of squares. Suppose a person standing on a square can only see diagonally[1]. For example, if the black circles in the diagram below represent people, the arrows indicate their direction of sight. These two people cannot see each other.



Problem: Determine the maximum number of people that can stand on the grid so that no two people can see each other, and determine where they should stand.



TECHNIQUE

[1]If you are familiar with the game *chess*, it might help to think of these people as bishops.

Name: ______________________________

4. Application of Graph Theory (continued)

c. You are organizing a ping-pong game night. Before the event, every participant (named $A, B, \ldots, H$) sends you a list of which people they want to play against.

$A : \{B, C, D, F, G\}$ $E : \{A, C, F, G, H\}$
$B : \{C, D, E, G, H\}$ $F : \{A, B, D, E\}$
$C : \{A, D, G, H\}$ $G : \{B, C, H\}$
$D : \{B, C, F, H\}$ $H : \{A, B, G\}$

Problem: Determine a schedule for the games so that two players compete against each other if and only if both players *want* to compete against each other. Each player can participate in one game at a time, but many games can happen simultaneously. Your schedule should minimize the total time required for the event, assuming every game lasts exactly one hour.

TECHNIQUE

Name: ______________________________

5. New Proof

a. Let G be a graph with a Hamiltonian path, and let S be any subset of $V(G)$. Prove that $G - S$ has at most $|S| + 1$ connected components. (100,000 points)

b. Let G be a connected graph, and let S be any subset of $V(G)$. Give an example to show that it is *not* true in general that $G - S$ must have at most $|S| + 1$ connected components. (30,000 points)

Name: ______________________________

6. New Proof 2

Let G be any simple graph. Show that $|E(G)| \geq \binom{\chi(G)}{2}$, where $\chi(G)$ is the chromatic number of G. Recall that the notation $\binom{a}{b}$ denotes the number of b-element subsets of an a-element set, so $\binom{a}{2} = a(a-1)/2$. (100,000 points)

Name: ______________________________

7. New Proof 3

a. For which values of n and m is $K_{n,m}$ planar? You do not need to justify your answer. (50,000 points)

b. For a simple graph G, the **complement** of G, written $\overline{G}$, is the simple graph with $V(\overline{G}) = V(G)$, and two vertices are connected by an edge in $\overline{G}$ if they are *not* connected by an edge in G. Prove that for any simple planar graph G with $|V(G)| \geq 11$, the graph $\overline{G}$ is *not* planar. (100,000 points)