Name: ____________________________

Student Number: ____________________________

Midterm Exam

Math 332, Fall 2016, Geoffrey Scott

DO NOT OPEN THIS BOOKLET UNTIL YOU ARE TOLD TO DO SO

Instructions:

- Put your name and student number on this page.
- You have 110 minutes to complete the exam. I will write the remaining time on the blackboard.
- You may ask me questions, but I cannot answer math questions or questions like "have I shown enough work?"
- I will give partial credit for some questions, so show your work.
- You may use the back of the pages as scratch paper. If you need to use the back of the pages to write your answer, make sure you write "*SEE BACK OF PAGE*" so that the grader knows where to look.
- You may leave early. Put your completed exam on the front table and have a nice evening.

Grades:

Question 1: ________ (out of 10)

Question 2: ________ (out of 10)

Question 3: ________ (out of 20)

Question 4: ________ (out of 6)

Question 5: ________ (out of 6)

Question 6: ________ (out of 10)

Total: ________ (out of 62)

Confusing Notation

Walk: A **walk** in a graph is a sequence of the form $v_0e_1v_1e_2v_2 \dots e_nv_n$, where the v_i are vertices and the e_i are edges, such that the endpoints of e_i are v_{i-1} and v_i. A walk is **closed** if $v_0 = v_n$.

Trail: A **trail** is a walk in which no edge occurs more than once.

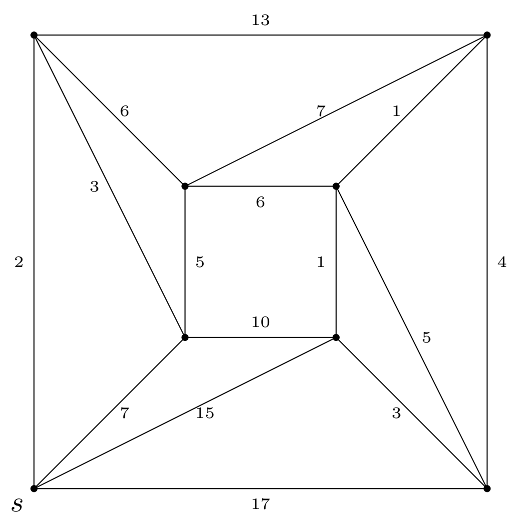
Path: A **path** is a trail in which no vertex occurs more than once, except that v_0 may equal v_n.

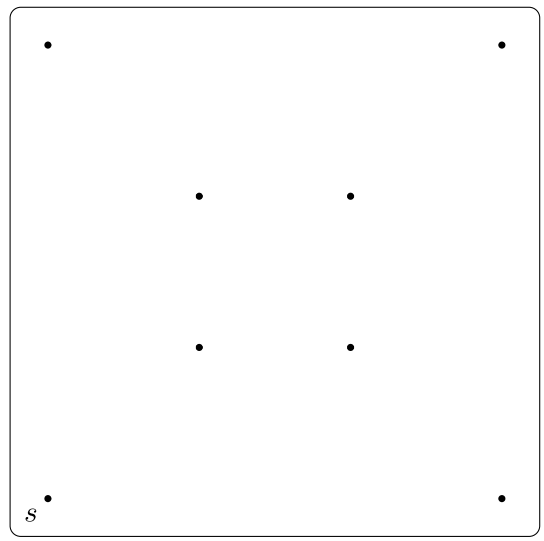
Circuit: A **circuit** is a closed trail.

Cycle: A **cycle** is a closed path.

1. Algorithms

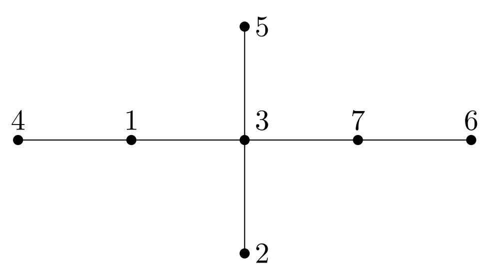
a. Suppose you apply Dijkstra's algorithm to the graph below using the vertex labelled s as your "start" vertex. Draw the first four edges that get added to your subgraph in the box labelled ANSWER, using the vertices provided. There is a step in Dijkstra's algorithm where you might need to make an arbitrary choice between which edge to add, resulting in multiple valid answers. If this happens, any valid answer will be given full credit. (5 points)





ANSWER

b. Write the Prüfer code for the tree shown below in the box labelled ANSWER. (5 points)



ANSWER

2. Proofs from Class

a. Prove that every tree with at least one vertex has exactly $|V(G)| - 1$ edges. If you need it, you may use the fact that every tree with ≥ 2 vertices has at least two leaves, and that every tournament has a Hamiltonian path. (6 points)

b. State, but do not prove, Hall's theorem. Be precise, as though you were writing a textbook! If you the phrase "Hall's condition," state what this is. (4 points)

3. Short Answer

Give an example of the following objects, or give a short (one or two sentences should suffice) explanation why no example exists.

a. A simple connected graph with 7 vertices and 6 edges that contains an odd number of distinct cycles. For this problem, two cycles are counted as being "the same" if they use the same set of vertices. (4 points)

b. A simple connected graph G with ≥ 3 vertices such that both G and $G-e$ have an Eulerian circuit for some $e \in E(G)$. (4 points)

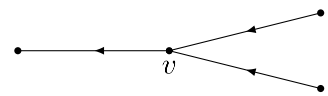
c. A simple connected graph G with ≥ 3 vertices such that both G and $G-e$ have a Hamiltonian cycle for some $e \in E(G)$. (4 points)

d. A simple graph with at least two vertices and degree sequence $(|V(G)|-1, |V(G)|-2, \ldots 3, 2, 1, 0)$. (4 points)

e. A simple connected graph G such that the maximum size of a matching in G is 2, and the minimum size of a vertex cover in G is 3. (4 points)

4. New Proof

In a directed graph, the **indegree** of a vertex v, written $d_i(v)$ is the number of edges incident to v that are directed *towards* v, and the **outdegree**, written $d_o(v)$ is the number of edges incident to v directed *away from* v. For example, the vertex v in the graph below has indegree 2 and outdegree 1.



Prove that for any directed graph G,

$$\sum_{v \in V(G)} d_i(v) = \sum_{v \in V(G)} d_o(v).$$

(6 points)

5. New Proof 2

Let e be an edge of K_n. Calculate the number of spanning trees of $K_n - e$. Explain how you got your answer. (6 points)

6. New Proof 3

Consider a deck of 52 cards where each card has an integer from 1 to 13 written on it, and each such number is written on exactly four cards[1]. Suppose I shuffle the cards and deal them face-down into 13 piles, each pile containing four cards. Prove that it is possible for me to examine each pile, then pick exactly one card from each pile so that in total I've picked exactly one card of each number. (10 points)

[1]This is just a "standard" deck of cards without jokers. The only purpose of this sentence is in case anyone is unfamiliar with what a "standard" deck of cards is